MATH 121 PRACTICE FINAL, DECEMBER 2014

Name:

Please answer each question in the space given. There is an extra page at the back of the exam for your scrap work, but that work will not be graded.

Please write neatly. If I can't read your answer, I can't give you credit.

Good luck!

(1) True or False.

(a) If B is a 6×6 matrix with characteristic polynomial $\lambda^3(\lambda - 3)^2(\lambda + 5)$, then rank(B) is at least 3.

(b) If D and E are matrices satisfying $DE = ED$ and D is diagonalizable (over the complex numbers), then E must be also diagonalizable (over the complex numbers).

(c) Let $T : V \to V$ be a linear operator. If the characteristic polynomial of T is equal to the minimal polynomial of T, then T is diagonalizable.

(2) Find a 3×3 matrix $D : \mathbb{R}^3 \to \mathbb{R}^3$ whose null space consists of all vectors of the form $\begin{pmatrix} r \\ r \\ r \end{pmatrix}$, and whose range consists of all vectors of the form $\begin{pmatrix} s+t \\ s-t \\ 2t \end{pmatrix}$ (Here r, s, t are arbitrary real numbers.)

(3) One of the ways we defined the determinant of a matrix was as the unique function from $Mat_{n\times n}(F) \to F$ that satisfies three properties. List those properties.

(4) Define the following:

Vector Space

Direct sum

(5) Fix a field F and let V be a vector space over F and $T \in \mathcal{L}(V, V)$ a linear operator. Fix two elements $a, b \in F$. Prove that $(T - aI)$ commutes with $(T - bI)$.

(6) Suppose $A \in Mat_{7\times 7}(\mathbb{C})$ is a matrix with characteristic polynomial $p(z) = (z-2)^2(z-3)^2(z-4)^3$. Suppose as well that $dim(Null(A - 2I)) = 1$, $dim(Null(A-3I)) = 2$, and $dim(Null(A-4I)) = 1$.

Find a matrix which is the Jordan canonical form of A. Is the matrix you found unique?

(7) Consider $B = \begin{pmatrix} 9 & k \\ k & 1 \end{pmatrix}$, where k is a real number.

(a) For which values of k is B invertible?

(b) For which values of k is B diagonalizable?

(c) Find the eigenvalues of the matrix B. Your answer should be in terms of k.

(8) Let T be a linear operator over C with characteristic polynomial $(z - \lambda_1) \cdots (z - \lambda_n)$, and let $f(z)$ be a polynomial. Prove that the characteristic polynomial of $f(T)$ is $(z - f(\lambda_1)) \cdots (z - f(\lambda_n))$.

(9) Let A be the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

Find the determinant, rank, characteristic polynomial, the eigenvalues and the eigenvectors of A. Find an invertible matrix Q such that $D = Q^{-1}AQ$. Compute D.

(10) Consider the linear transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $T(f)(x) = f(x) + f(-x)$. Describe the null space of T and the range of T as subspaces of $\mathbb{R}[x]$.

(11) The Fibonacci sequence is defined by the recursive formula $a_1 = 1, a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Compute a non-recursive formula for a_n as a function of n. *Hint*: use diagonalization to compute A^n for a suitable matrix A.

(12) $\mathbb{C}$ can be considered as a vector space over $\mathbb{C}$ and over $\mathbb{R}$.

a) What is the dimension of $\mathbb{C}$ when considered a vector space over $\mathbb{C}$? Over $\mathbb{R}$?

b) Prove that $S : \mathbb{C} \to \mathbb{C}$ given by $S(z) = \overline{z}$ is additive but not $\mathbb{C}$-linear. Show that it is $\mathbb{R}$-linear and compute its matrix with respect to the ordered basis $\{1, i\}$.

Page for scratchwork. Nothing on this page will be graded.